Center Of A Closed Convex Polytope: A New Definition

V. S. Patwardhan[1]

Abstract

A closed convex polytope in n dimensions is defined by the usual linear inequality constraints specified as $Ax \leq b$. Let P be a strictly interior point within the closed polytope. If we draw a line through P which is parallel to the i^{th} axis, it will intersect the polytope boundary at two points. Let S_i be the line segment connecting these two points. A new center, termed here as the bisection center, or the BI center, is defined as that interior point which bisects the segments S_i for all values of i, i.e. i = 1 to n. The existence and uniqueness of such a center for the general case has been shown. An algorithm is proposed for calculating the BI center iteratively. Preliminary computational results are presented for two simple polytopes for illustration. The algorithm is expected to be computationally efficient, particularly for highly sparse matrices, since (a) it can take full advantage of the sparse structure of the A matrix, (b) it involves no matrix inversion, and (c) the A matrix remains unchanged during calculations. Application of these ideas to larger size problems, including linear programming problems will be presented in a later publication.

1. Introduction

Defining and calculating the center of a closed convex polytope is a problem that has attracted interest for a long time. Many definitions have been used for the center, based on different considerations such as the center of mass of the entire polytope, the centroid i.e. the center of mass of all vertices, the center of the largest inscribed sphere (or an ellipsoid), the center of the smallest sphere (or ellipsoid) which includes the polytope, the analytical center, the weighted projection center, center based on orthogonal projections on to polytope faces etc. All these definitions lead to different points as centers. The computation of these centers involves different degrees of computational effort. Some of these and others have been summarised earlier (for example, see A. Moretti [1], K. G. Murty [2]). One of the motivations for these centers has been the fact that most interior point algorithms for solving linear programming problems involve a centering step at periodic intervals. Therefore any method which leads to efficient calculation of a uniquely defined center may become useful in interior point linear programming algorithms.

In this paper, we present the definition of a new center termed here as the BI center, some discussion on the existence and uniqueness of the BI center and computational results obtained for a couple of simple polytopes.

2. Definition of the BI center

The BI center is based on the concept of the bisection of line segments within the polytope, which

[1] Independent researcher. Formerly, Scientist G, National Chemical Laboratory, Pune 411008, India. Email : vspatw@gmail.com

are drawn from any interior point and are parallel to one of the coordinate axes. Let us start with a two dimensional example for illustrating the concept. Figure 1 shows a two dimensional triangle as a simple example of a closed convex polytope. The figure shows an interior point P and a line in the x direction drawn through P. The line intersects the triangle boundary at P_1 and P_2. The point Q is the midpoint of the line segment P_1P_2, and bisects it. It is obvious that if P is moved along the line segment P_1P_2, the corresponding point Q does not change. However, if P is moved anywhere else, then Point Q will move as well. Thus the point Q depends only on the y-coordinate of P. By moving P over the entire triangle (i.e. the entire interior region of the polytope), we get a locus of the point Q, which is shown by the line L_x in Figure 1. The subscript x in L_x implies that it is the locus of midpoints of line segments parallel to the x-axis, which are contained in the polytope. In other words, any point on L_x bisects the line segment drawn through that point in the x-direction.

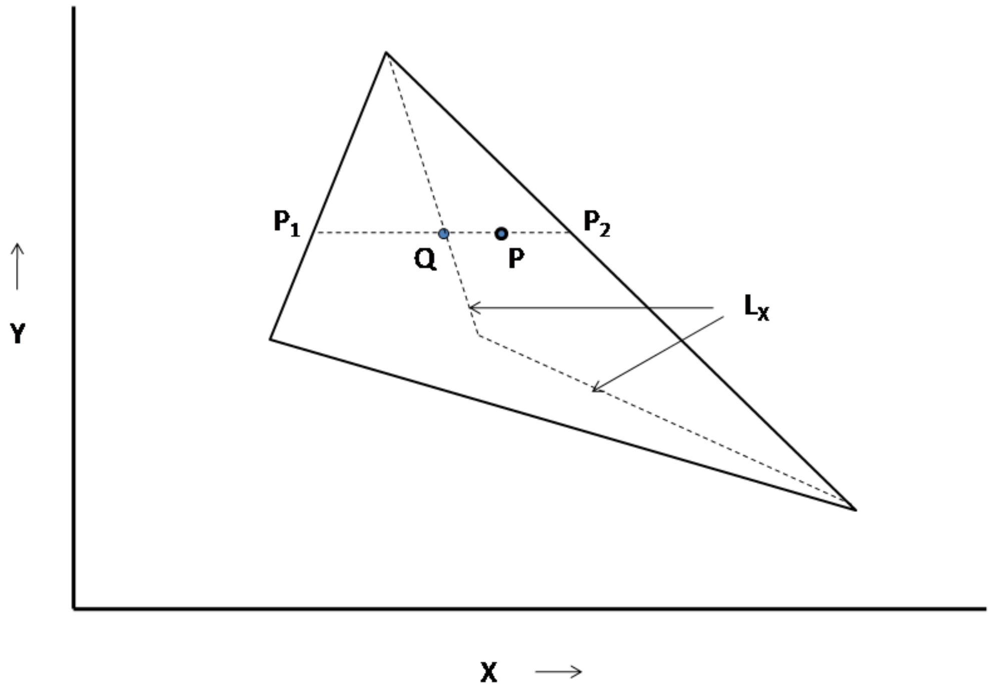

Figure 1. Locus of the bisector of line segments in the X direction

We can follow a similar procedure, by drawing line segments in the y direction in the same triangle, to get a different line, shown as L_y in Figure 2. Any point on L_y bisects the line segment drawn through that point in the y-direction. The line L_x is also shown in Figure 2. Point B shown in this figure is the intersection of lines L_x and L_y, and is the only point in the polytope which bisects both the line segments (i.e. a line segment in the x direction, and another one in the y direction) drawn through it, and represents the BI center of the triangle. The BI center for more complex cases can be shown in a similar manner. Figure 3, for example, shows the BI center for a two dimensional convex polytope with five sides, i.e. a pentagon.

We can make a few important observations from Figure 2. The lines L_x and L_y are not entirely straight lines, but consist of straight line segments. They show a sharp turn at specific points, corresponding

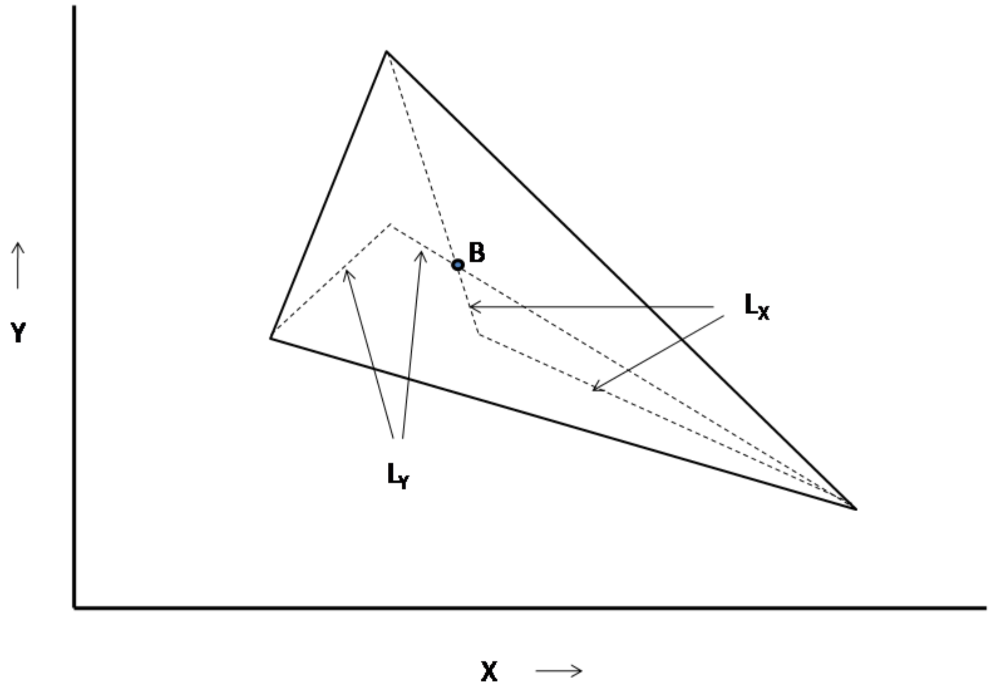

Figure 2. The BI center, as the intersection of lines L_x and L_y

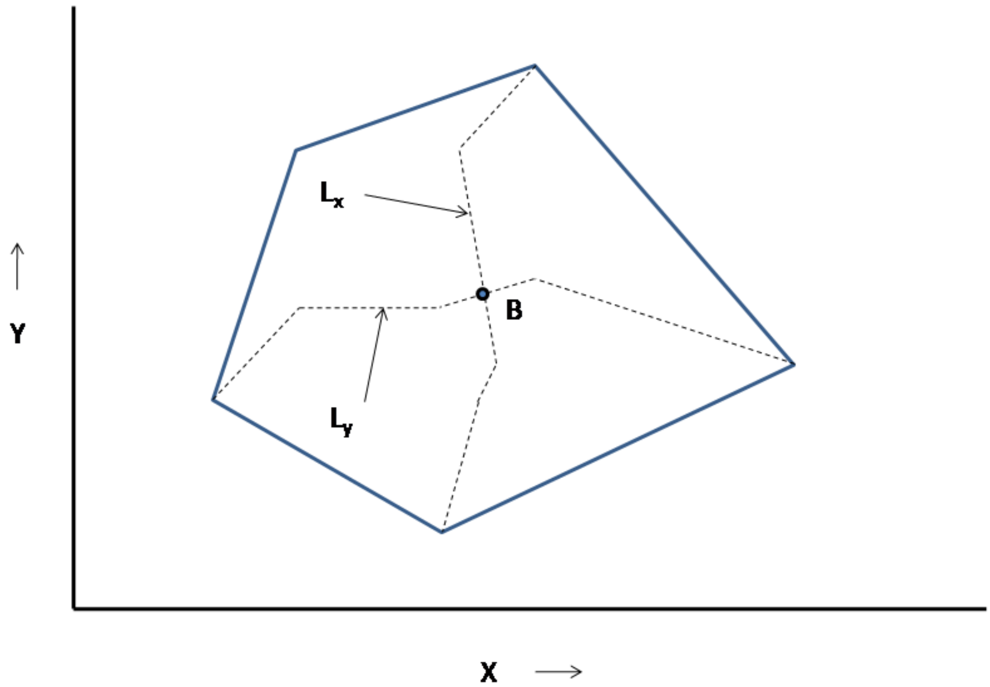

Figure 3. The BI center and lines L_x and L_y for a pentagon

to the vertices of the triangle. Thus L_x and L_y are continuous but not differentiable. Secondly, the length of the two line segments drawn through point B (which are bisected by point B) can be widely different, for example, if one of the angles of the triangle is very small, leading to a sharp, wedge-like shape.

The concept of the BI center can be easily generalised for higher dimensions. For example, let us consider a convex polytope in three dimensions. Imagine a large number of line segments, parallel to the X axis, drawn inside this polytope. The set of midpoints of these line segments defines a two dimensional surface which can be denoted as S_x. Similarly, by considering line segments drawn in the Y and Z directions, we can get two more two dimensional surfaces, i.e. S_y and S_z. These three surfaces intersect at a single point, in general, which is the BI center of the polytope.

For a closed convex polytope in an n-dimensional space, this procedure gives us n subspaces, each of which is (n-1) dimensional. They intersect at a single point, in general, which is the BI center of the closed convex polytope. The BI center is unique, in general, for an n-dimensional polytope, as shown in a later section.

3. Characterisation of the BI center

The geometrical discussion given so far defines the BI center. We will later present calculations where we start from an interior point and approach the BI center iteratively. Before doing that, let us first see how we can characterise the BI center, so that we can determine how far or how close any given interior point is, from the BI center. We define a function here which becomes equal to 1 at the BI center and is equal to zero at the polytope boundary.

Figure 1 shows the point Q which bisects the line segment P_1P_2 drawn through point P, and is parallel to the x axis. We define a function F_x at the point P as

$$F_x = 4 * PP_1 * PP_2/(P_1P_2)^2 \tag{1}$$

We define f_x as the fraction PP_1/P_1P_2. Since $P_1P_2 = PP_1 + PP_2$, we can Express F_x as

$$F_x = 4 * f_x * (1 - f_x) \tag{2}$$

It is obvious that f_x goes from 0 to 1 as P moves from P_1 to P_2. Moreover, F_x is zero at both P_1 and P_2, and goes through a single maximum of $F_x = 1$ at $f_x = 0.5$ which corresponds to Q, the midpoint of P_1P_2. It is also obvious that the second derivative of F_x w.r.t. x is negative everywhere along P_1P_2. At any interior point P within the polytope, F_x has some value between 0 and 1 depending on the location of P. The factor of 4 has been used just to make F_x equal to 1 at Q. A similar function F_y can be defined based on the line segment through P drawn in the y direction. We now define a composite function F as

$$F = F_x * F_y \tag{3}$$

Referring to Figure 2, it is obvious that F is zero at the triangle boundary, is equal to 1 at the BI center, and has some value between 0 and 1 everywhere else. For a closed convex polytope in n dimensions, the composite function at any interior point P can be written as

$$F = \prod_{i=1\ to\ n} F_{x_i} \tag{4}$$

where F_{xi} is based on the line segment drawn through P, in a direction parallel to the i^{th} coordinate axis. Even in this case, F becomes 1 only at the BI center, and is zero at the polytope boundary.

4. Uniqueness of the BI center

To prove the uniqueness of the BI center in an n-dimensional closed convex polytope, let us first show that the first derivative of F_{xi} along any straight line segment, whose end points lie on the polytope boundary, is zero only at one point along that line segment.

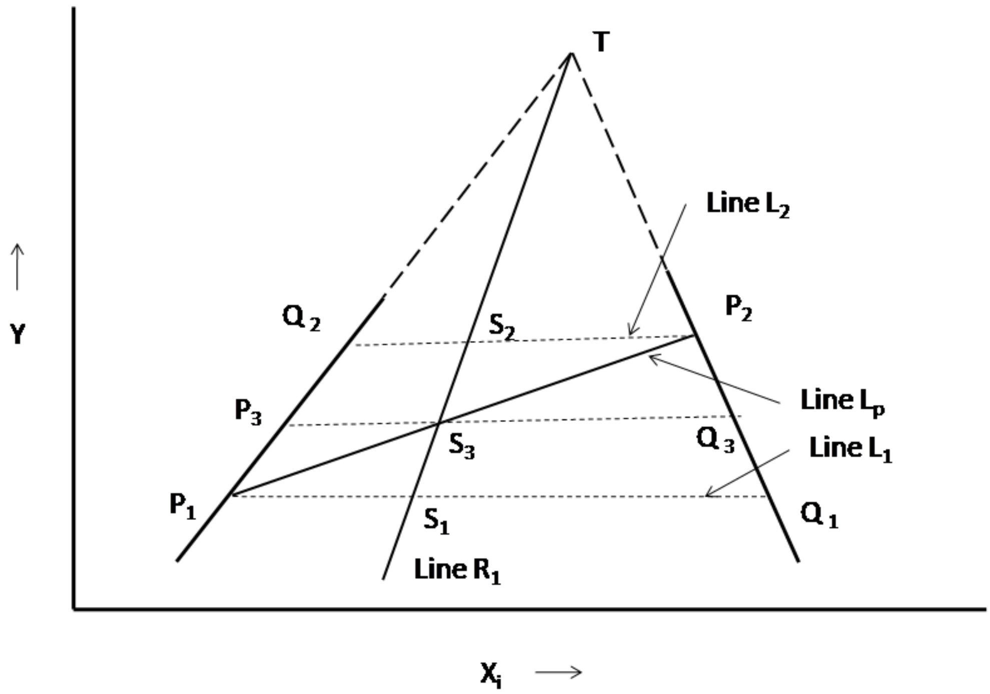

Figure 4. A cross section of the polytope in two dimensions

Let us consider the function F_{xi}. Let P be a strictly interior point. Consider an arbitrary straight line L_p drawn through P, which intersects the polytope boundary at points P_1 and P_2. Consider two lines L_1 and L_2, which are drawn through P_1 and P_2 respectively in a direction parallel to the x_i axis. Now consider the intersection of the polytope and the two dimensional plane defined by these three lines. This intersection is a two dimensional closed convex polygon, and is shown partly in Figure 4 with bold continuous lines. First consider the case where the points P_1 and Q_2 lie on the same side of the polygon (i.e. the same face of the polytope), and so do Q_1 and P_2. (The case where they lie on different sides of the polygon is considered later). T is the intersection of lines P_1Q_2 and P_2Q_1. The figure also shows a line R_1 drawn through T, along with its three intersections S_1, S_2 and S_3. Triangles TP_1S_1 and TQ_2S_2 are similar, and so are triangles TP_1Q_1 and TQ_2P_2. Therefore the ratios Q_2S_2 / Q_2P_2 and P_1S_1 / P_1Q_1 are equal, and so are F_{xi} values at any point along the line R_1. Thus, F_{xi} values at S_1, S_2

and S_3 Are equal. (It should be borne in mind that F_{xi} value at S3 is determined by the ratio P_3S_3 / P_3Q_3, and not P_1S_3 / P_1P_2).

Let us consider the variation of function F_{xi} along the line L_1. F_{xi} and its derivative are given by

$$F_{xi} = 4\ x\,(Q_{1x} - x)/Q_{1x}^2 \tag{5}$$

$$\frac{dF_{xi}}{dx} = \frac{4}{Q_{1x}} - \frac{8x}{Q_{1x}^2} \tag{6}$$

where x is the distance P_1S_1 and Q_{1x} is the distance P_1Q_1. Equations (5) and (6) show that F_{xi} is 0 for x = 0 and x = Q_{1x}, and dF_{xi}/dx is zero only at one point, i.e. at x = $Q_{1x}/2$.

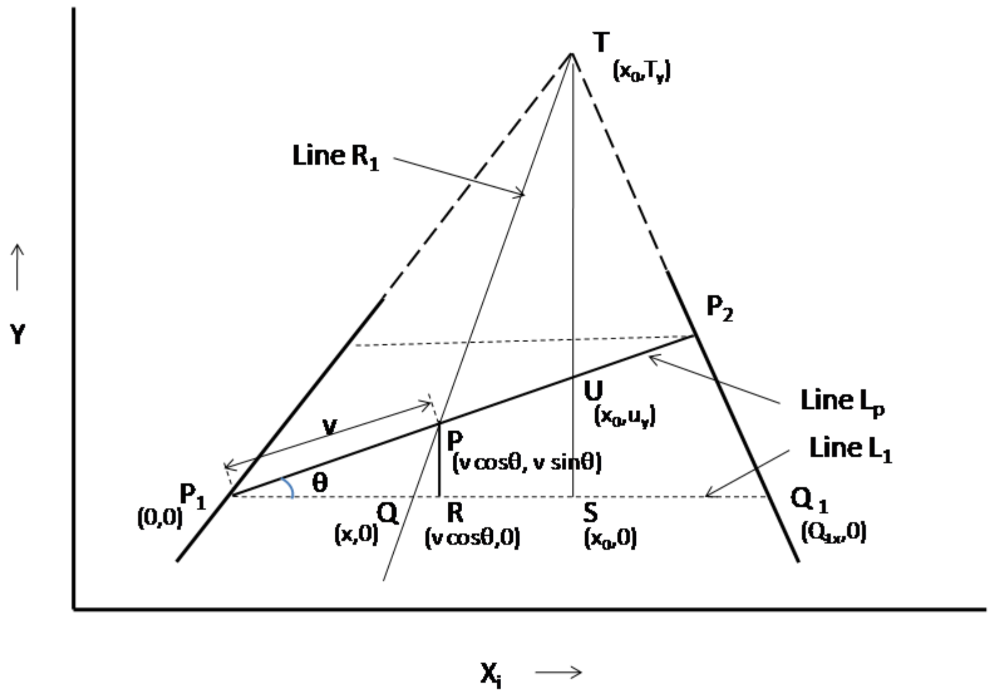

Figure 5. Variation of F_{xi} along the line L_p

Let us now consider the variation of function F_{xi} along the line L_p. Figure 5 shows a general point P on the line L_p. The distance P_1P is denoted as v. Q is the intersection of lines TP and P_1Q_1. Lines PR and TS are drawn perpendicular to line P_1Q_1. The coordinates of all these points are also indicated in Figure 5. Now, similar triangles PRQ and TSQ give

$$\frac{v\ cos\theta - x}{v\ sin\theta} = \frac{x_0 - x}{T_y} \tag{7}$$

Algebraic manipulation gives

$$\frac{1}{v} = \frac{K}{x} + \frac{\sin\theta}{T_y} \tag{8}$$

where the constant K is given by

$$K = \cos\theta - \frac{x_0 \sin\theta}{T_y} \tag{9}$$

From Figure 5, it is obvious that $T_y > U_y$. Since $\tan\theta = U_y/x_0$, we can write

$$K = \cos\theta \left(1 - U_y/T_y\right) \tag{10}$$

Thus, K is always positive. Differentiation of Eqn (8) followed by algebraic manipulation gives

$$\frac{dv}{dx} = K\,\frac{v^2}{x^2} \tag{11}$$

This implies that $dv/dx > 0$. This is consistent with the observation from Figure 5 that, as point P moves along the line L_p, point Q moves in the same direction too. The derivative of F_{xi} along the direction L_p can be written as

$$\frac{dF_{xi}}{dv} = \frac{dF_{xi}}{dx} \Big/ \frac{dv}{dx} \tag{12}$$

We have already seen that $dv/dx > 0$, and dF_{xi}/dx vanishes only at a single value of x. Therefore we can conclude that dF_{xi}/dv also vanishes only at a single value of v. This is valid for any coordinate axis x_i, i = 1 to n.

We are interested in showing the uniqueness of the BI center. Suppose it is not unique. In that case there will be at least two BI centers, i.e. B_1 and B_2. Both these points would have F = 1, and at both these points, dF_{xi}/dv would vanish, for all values of i. Let us consider the line joining B_1 and B_2, extended in both directions to intersect the polytope boundary at P_1 and P_2. Thus, dF_{xi}/dv must vanish at least at two different points along the line segment P_1P_2. This clearly contradicts the conclusion stated above, and shows that the BI center is indeed unique.

Case where the points Q_1 and P_2 lie on different sides of the polygon: In the discussion above, it was assumed that points Q_1 and P_2 lie on the same side of the polygon. Figure 6 (which is a simplified Figure 5) shows several lines inside triangle $P_1Q_1P_2$. Each of these lines passes through Point T, and its intersections along sides P_1P_2 and P_1Q_1 represent two points with equal values of F_{xi}. The fact that these lines do not intersect among themselves, is equivalent to saying that $dv/dx > 0$, as argued above. Now let us consider a case where points Q_1 and P_2 do not lie on the same face (i.e. side) of the polygon. This situation is shown in Figure 7. Here the point P_2 does not lie on the same face (i.e.side) of the polygon as point Q_1. The two polygon sides corresponding to points P_2 and Q_1 meet at point R. Line L_2 is a line drawn through R and is parallel to the X_i axis. In order to mark points having equal F_{xi} along lines L_1 and L_p we need to consider two regions. In the region between lines L_1 and L_2, we follow the same procedure as before, using point T to draw a fan of lines. In the region between lines L_2 and L_3, we now have to use point Z, which is the intersection of the two polygon

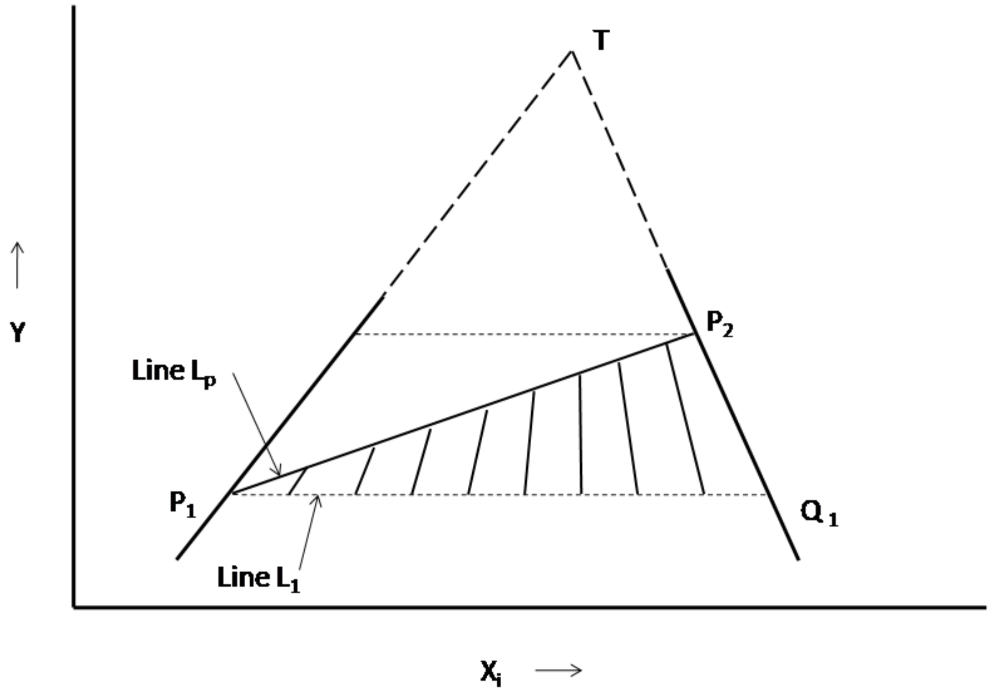

Figure 6. Corresponding points (with equal F_{xi}) along lines L_1 and L_p

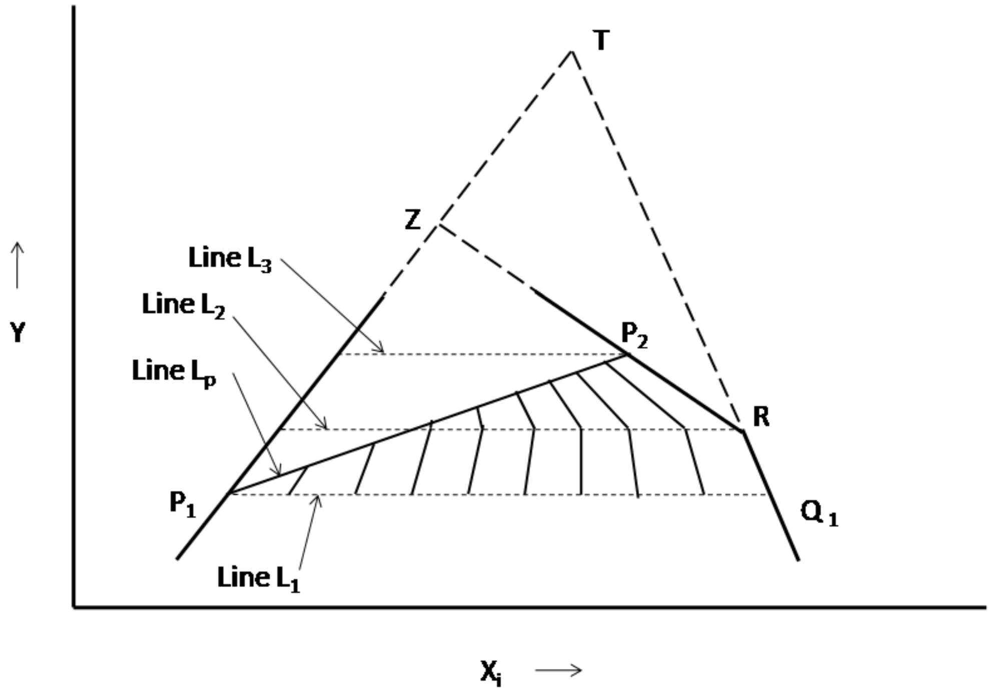

Figure 7. Corresponding points (with equal F_{xi}) along lines L_1 and L_p for a complex case

faces (i.e. sides) corresponding to points Q_1 and P_2, to draw a fan of lines. The resulting equal-F_{xi} lines are shown in Figure 7. Some of these lines consist of two line segments. However these composite solid lines drawn inside the region $P_1Q_1RP_2$ do not intersect among themselves. This is equivalent to saying that dv/dx > 0, and therefore dF_{xi}/dv vanishes only for one value of v (i.e. only at one single point along line L_p).

5. Illustrative calculations

Let us consider a small problem in two dimensions as an illustration. The polytope is defined by the constraints given below.

Example 1:

$$0.8\,x - y \leq 2$$

$$x + 2\,y \leq 18$$

$$5\,x + 3\,y \geq 30$$

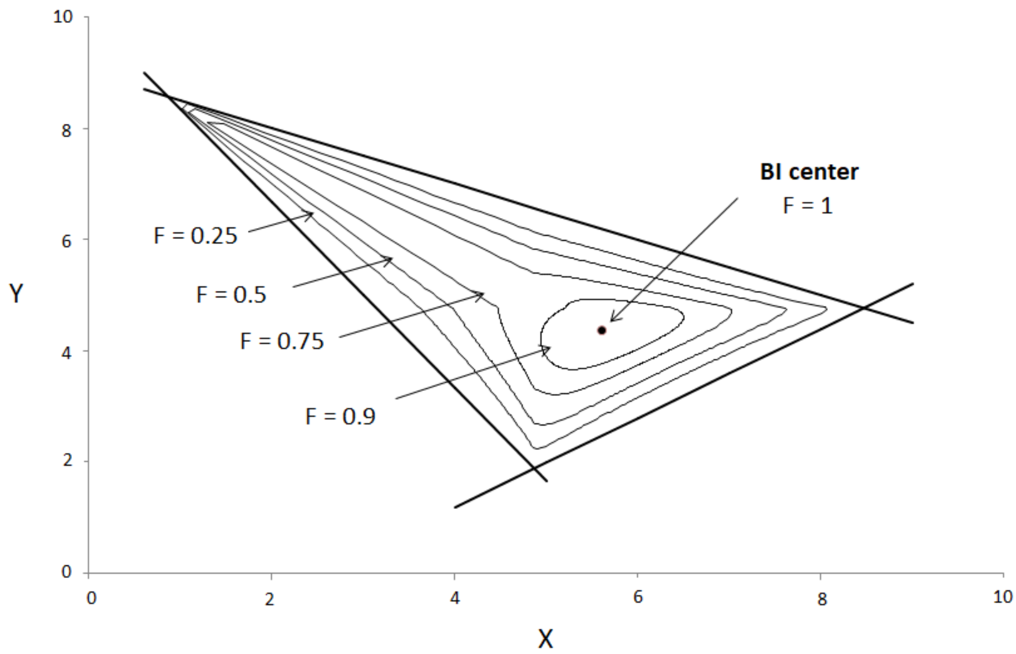

Figure 8. Contour lines for F for example 1

Figure 8 shows the polytope (a triangle in this case), contour lines for F, and the BI center. The contour lines get elongated in the direction of the vertices of the triangle, and become smoother as we move towards the BI center.

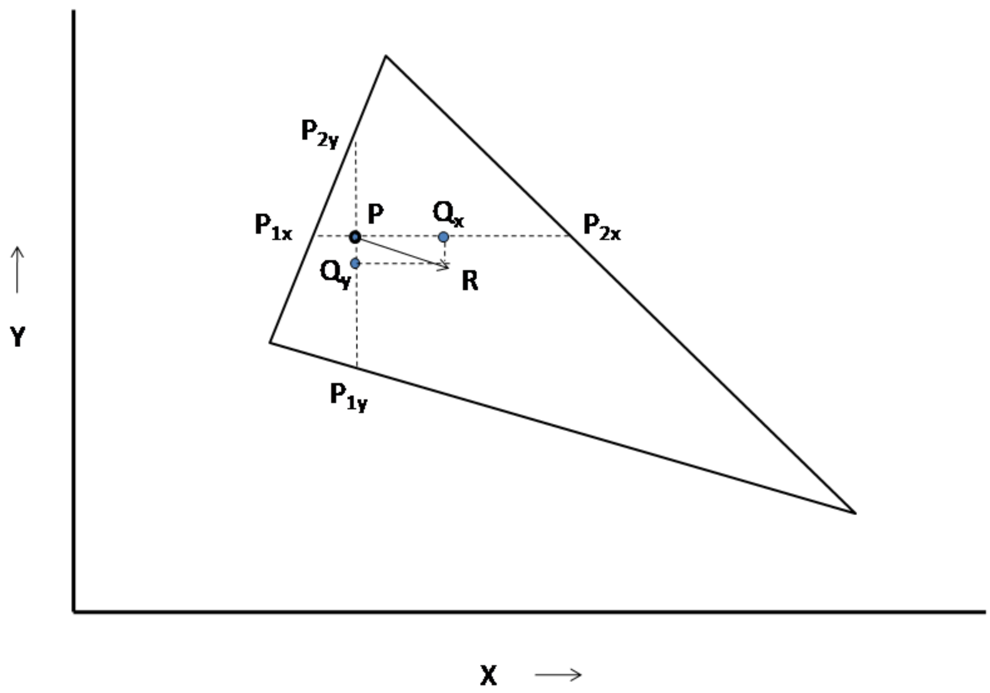

Figure 9. One step in the calculation of BI center

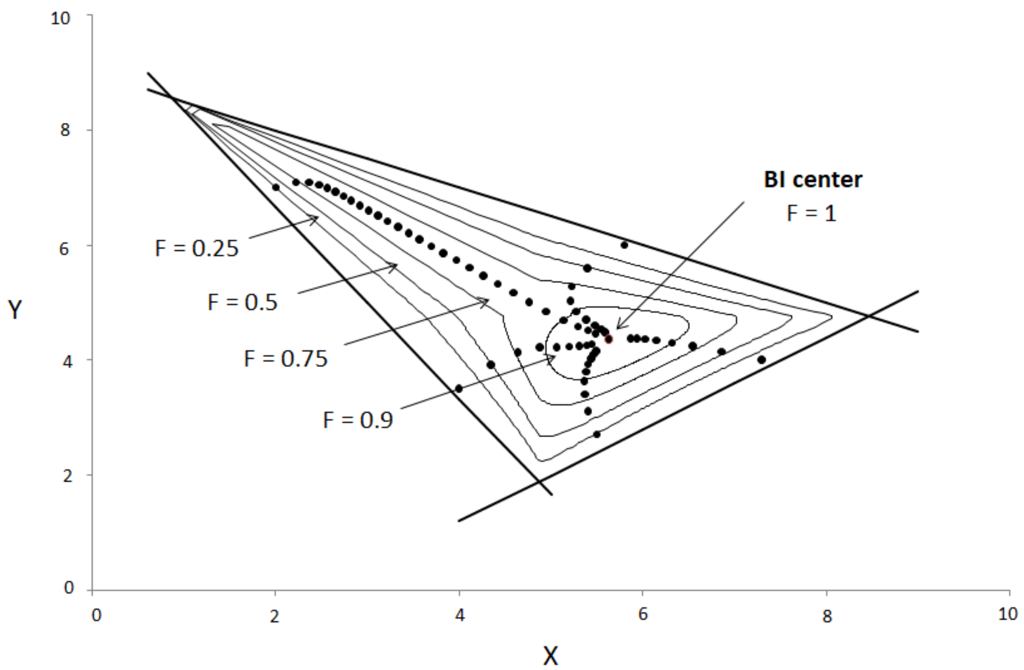

Figure 10. Approach to the BI center

6. Calculation of the BI center: preliminary results

The calculation of the BI center is done iteratively, starting from a strictly interior point. It is assumed here that such a point is known. (If not, one can be located as explained later.) Figure 9 shows one step of the procedure graphically. The interior point P is the starting point. Q_x is the midpoint of the line segment lying inside the polytope, drawn in the x direction. Similarly, Q_y is the midpoint of the line segment lying inside the polytope, drawn in the y direction. The vector addition of PQ_x and PQ_y gives the vector PR. If we take PR itself as the step, R would be the point reached after one step. However, we have used a conservative step size of PR/2 here, so that we reach the point S, which is the midpoint of line PR, after one step. (The point S may be infeasible, as may happen sometimes. In that case, the step size has to be reduced further, so that S remains strictly feasible.) Now S can be used as the starting point for the next step. This procedure, used iteratively, approaches the BI center. The approach to the BI center is characterised by the increasing F values which approach 1 closely. The iterative approach to the BI center (for Example 1) from different starting points, close to the polytope boundary, is shown in Figure 10, which also shows the contour lines of F.

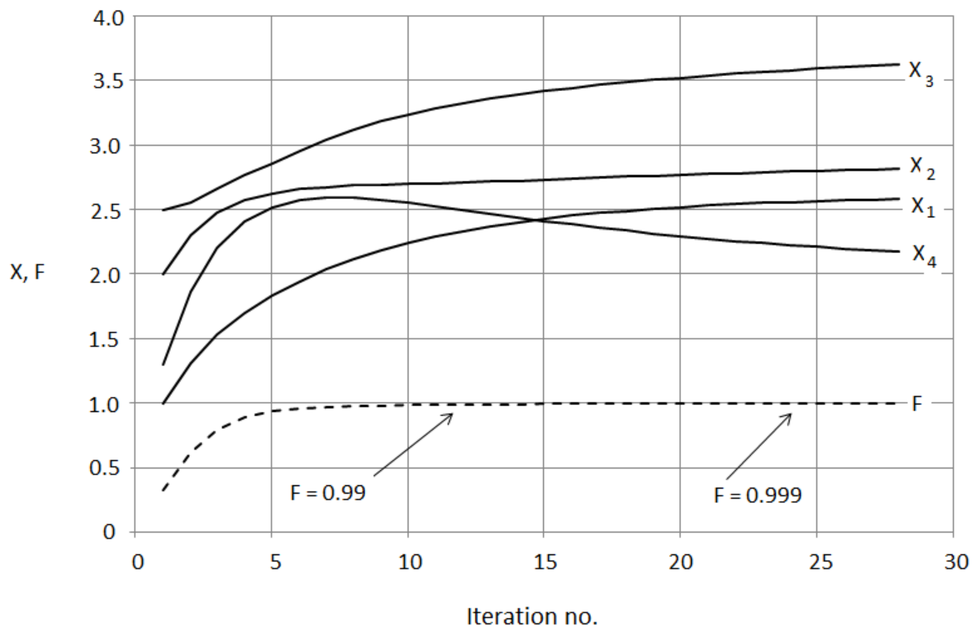

Figure 11. Approach of X values and F to their final values for Example 2

Let us consider a larger problem in four dimensions as given below.

Example 2:

$$x_1 + x_2 - x_3 + x_4 \leq 8$$

$$x_1 + 0.5\, x_2 - x_3 \leq 3$$

$$0.5\, x_1 - 2\, x_2 + x_3 \leq 2$$

$$-x_1 + 0.5\,x_2 - 0.5\,x_4 \leq 3$$

$$x_1 + 3\,x_2 + 1.5\,x_3 + 2\,x_4 \leq 25$$

$$x_1, x_2, x_3, x_4 \geq 0$$

This example has five constraints defining the polytope, and in addition, all four variables are nonnegative, which is equivalent to four more constraints. If we apply the centering algorithm to this polytope, starting from the arbitrary point (1, 2, 2.5, 1.3) we approach the BI center iteratively as shown in Figure 11 and Table 1. The values of X_i (i = 1 to 4) are seen to approach the BI center asymptotically. The function F becomes 0.99 after 12 iterations, and 0.999 after 24 iterations. In other words, we can approach the BI center very closely in a reasonable number of iterations.

Iter. No.	X_1	X_2	X_3	X_4	F
1	1.00	2.00	2.50	1.30	0.329
2	1.31	2.30	2.56	1.87	0.616
3	1.54	2.47	2.66	2.21	0.798
4	1.71	2.57	2.77	2.41	0.895
5	1.84	2.63	2.86	2.52	0.938
6	1.95	2.66	2.96	2.58	0.960
7	2.04	2.67	3.05	2.60	0.971
8	2.12	2.68	3.12	2.60	0.978
9	2.19	2.69	3.19	2.58	0.983
10	2.25	2.70	3.24	2.56	0.987
11	2.30	2.70	3.29	2.53	0.989
12	2.34	2.71	3.33	2.50	0.991
13	2.38	2.72	3.37	2.47	0.993
14	2.41	2.72	3.40	2.44	0.994
15	2.44	2.73	3.42	2.42	0.995
16	2.46	2.74	3.45	2.39	0.996
17	2.48	2.74	3.47	2.36	0.996
19	2.51	2.76	3.51	2.32	0.997
20	2.53	2.77	3.53	2.30	0.997
21	2.54	2.77	3.54	2.28	0.998
22	2.55	2.78	3.56	2.26	0.998
23	2.56	2.79	3.57	2.24	0.998
24	2.56	2.79	3.58	2.23	0.999

Table 1. Iterative approach to the BI center for Example 2.

7. Discussion

The main step in the iterative calculation of the BI center is shown in Figure 9. The calculation of Q_x from known P involves travelling in a direction parallel to the x axis, and calculating the intersections with all constraints defining the polytope. Q_y is calculated similarly. For an n dimensional polytope, the distance travelled in the $+X_k$ direction, from point P, before hitting constraint i is d_{ik+}, given by

$$d_{ik+} = \left(b_i - \sum_j a_{ij}\, P_j \right) \Big/ a_{ik} \qquad for\ all\ k\ where\ a_{ik} > 0 \tag{13}$$

And that in the -X_k direction is given by

$$d_{ik-} = \left(b_i - \sum_j a_{ij}\, P_j\right) \Big/ a_{ik} \qquad for\ all\ k\ where\ a_{ik} < 0 \tag{14}$$

The distance d_{ik} from P to the midpoint Q_{xk} is given by

$$d_{ik} = [min(d_{ik+}) + max(d_{ik-})]/2 \tag{15}$$

The direction for the iterative step for approaching the BI center is the vector of all d_k values. The quantity inside the bracket in equations (13) and (14) is positive since P is an interior point. The calculation of d_{ik} thus involves just division with nonzero elements of the A matrix. If A is large and sparse, then these calculations take full advantage of the sparsity. The A matrix does not change during these calculations, and there is no matrix inversion involved. If the number of iterations required is not too large, then the BI center can be approached in a computationally efficient manner. Computations with larger problems are in progress.

The calculation of the BI center starts from an interior point. It has been assumed here that an interior point is known. However, if one is not known, then techniques such as the introduction of an artificial variable followed by a procedure to reduce it to zero have to be used. This technique is well known in linear programming.

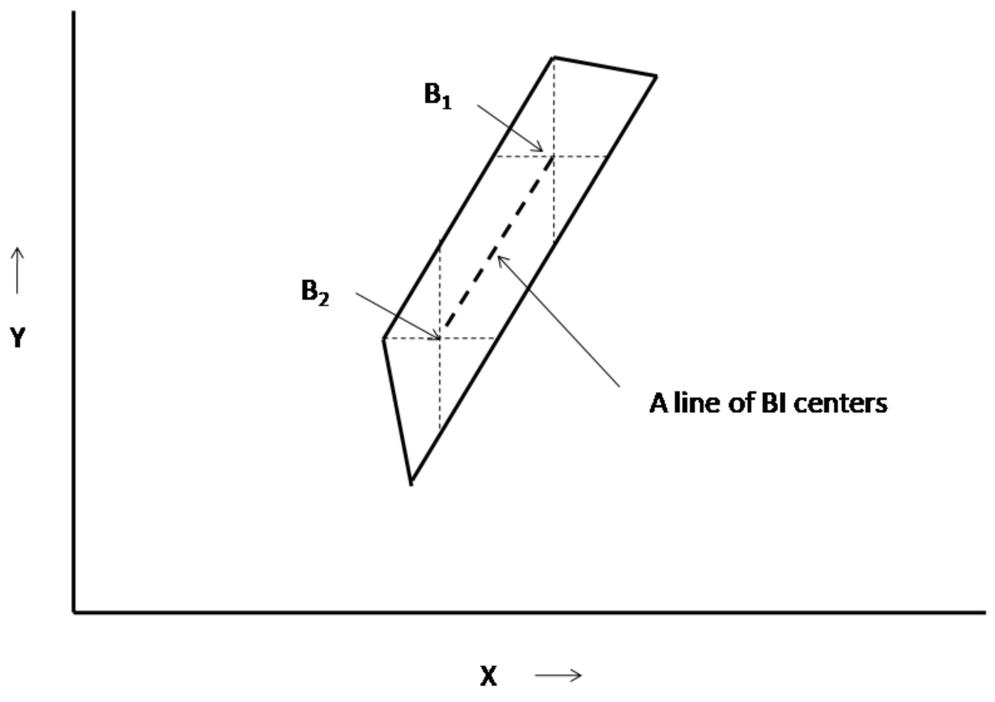

Figure 12. A degenerate case where the BI center is not unique

8. Degeneracy of the BI center

In this paper it has been shown that the BI center is unique in general. However there are cases where this is not true. Figure 12 shows a two dimensional example, where two sides of the polygon are parallel to each other. Points B_1 and B_2 are BI centers as both of them bisect the horizontal and vertical (dotted) lines passing through them. Moreover, any point along the line B_1B_2 is also a BI center. This degeneracy arises essentially because of the parallel sides of the polygon. Similar examples can be constructed in higher dimensions. This kind of degeneracy is known for centers defined differently, such as the well known Chebyshev center.

9. Conclusions

A new definition for the center of a closed convex polytope in n dimensions has been proposed. The newly defined center is termed as the BI center. The existence and uniqueness of the BI center has been shown for the general case. An iterative algorithm has been proposed for calculating the position of the center, starting from any interior point. The convergence characteristics of this algorithm are illustrated with computations for two small polytopes. It has been shown that special conditions can lead to degeneracy and multiplicity of the BI center. Work is in progress which is aimed at the application of these ideas to larger size problems, including linear programming problems.

10. References

[1] A. C. Moretti, *A weighted projection centering method*, Computational and Applied Mathematics, **22** (2003), pp 19–36

[2] K. G. Murty, *Optimization for Decision Making*, Springer, New York (2010)